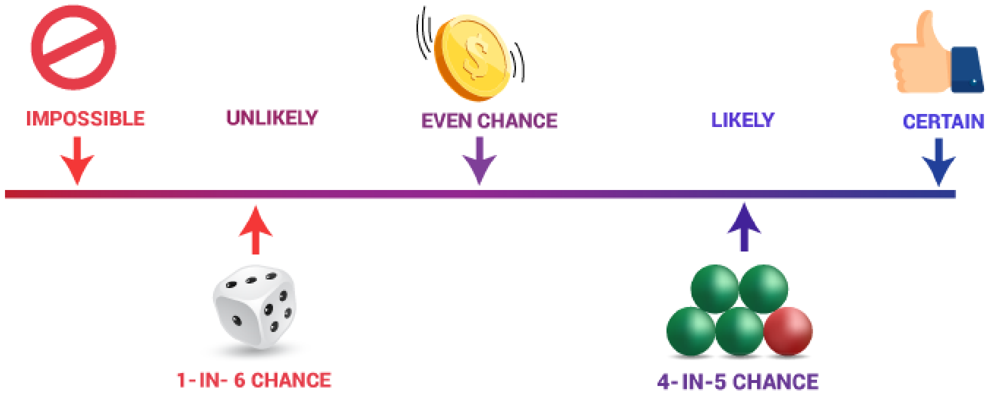
Probability



- **Probability of an Event**
- Methods of Assigning Probability
- Event Operations and Laws of Probability
- Conditional Probability and Independence
- Bayes’ Theorem

Probability of an Event (1)

- Probabilities express the *chance of events* that *cannot be predicted with certainty*.
- The probability of an event is viewed as a *numerical measure of the chance* that the event will occur.
- The probability of an event is naturally relevant to situations where the outcome of an experiment or observation *exhibits variation*.

> An **experiment** is the process of observing a phenomenon that has variation in its outcomes.

- Essential *terminologies*

> The **sample space** associated with an experiment is the collection of all possible distinct outcomes of the experiment.
>
> Each outcome is called an **elementary outcome,** a **simple event,** or an **element of the sample space.**
>
> An **event** is the set of elementary outcomes possessing a designated feature.

- We denotes

  The *sample space* by $\boldsymbol{S}$

  The *elementary outcomes* by $\boldsymbol{e_1}, \boldsymbol{e_2}, \boldsymbol{e_3}$, . . .

  Events by *A*, *B*, and so on.

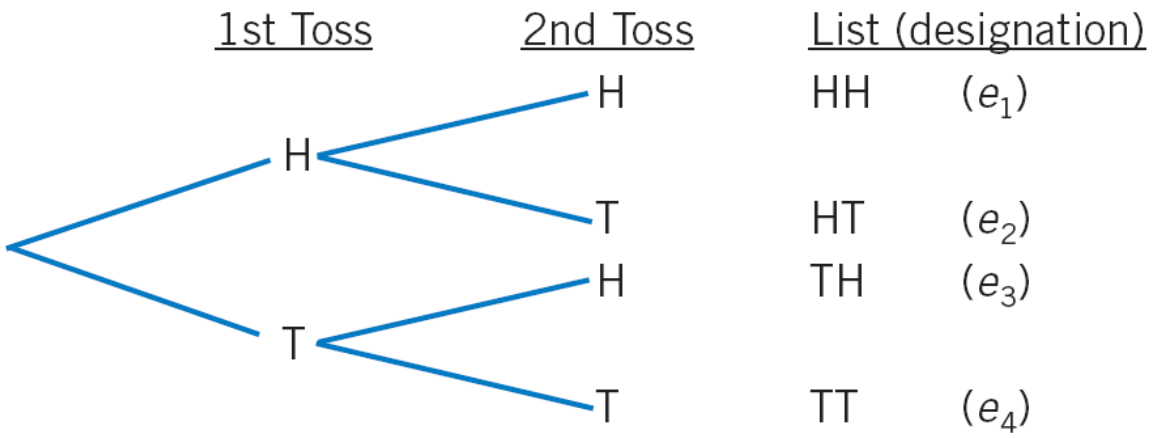
Probability of an Event (2)

An event *A* occurs when any one of the elementary outcomes in *A* occurs.

Example: Toss a coin twice and record the outcome head (H) or tail (T) for each toss. Let *A* denote the event of getting *exactly one head* and *B* the event of getting *no heads at all*.

- List the sample space and give the compositions of *A* and *B*.

❖ The elementary outcomes can be conveniently identified by means of a **tree diagram**.



$S = \{HH, HT, TH, TT\}$ or
$S = \{e_1, e_2. e_3, e_4\}$
$A = \{e_2. e_3\}$
$B = \{e_4\}$

Probability of an Event (3)

The **probability of an event** is a numerical value that represents the proportion of times the event is expected to occur when the experiment is repeated many times under identical conditions.

The probability of event A is denoted by $P(A)$.

- The probability of an event is the ***sum*** of the probabilities assigned to all the elementary outcomes contained in the event.
- The sum of probabilities of all the elements of S ***<u>must</u>*** be 1.
- In General:

Probability must satisfy:

1. $0 \leq P(A) \leq 1$ for all events A
2. $P(A) = \sum_{\text{all } e \text{ in } A} P(e)$
3. $P(S) = \sum_{\text{all } e \text{ in } S} P(e) = 1$

Probability of an Event (4)

Example: Construct a *sample space* and/or identify events for each of the following experiments.

a) Someone claims to be able to taste the difference between the same brand of *bottled*, *tap*, and *canned* draft beer. A glass of each is poured and given to the subject in an unknown order. The subject is asked to identify the contents of each glass. The number of correct identifications will be recorded.

Answer: $S = \{0, 1, 3\}$

Note: The number of correct identifications cannot be 2 because whenever any two are correctly identified, the third one cannot be wrong.

b) Not more than one correct identification.

Answer: $S = \{0, 1\}$

c) Record the number of traffic fatalities in a Norway next year.

Answer: $S = \{0, 1, 2, \dots\}$

d) Less accidents than last year fatalities. (Note: last year's value was 145).

Answer: $S = \{0, 1, 2, \dots, 144\}$

e) Observe the length of time a new digital video recorder will continue to work satisfactorily without service.

Answer: Denoting t = duration of satisfactory work, say in days, $S = \{t : t \geq 0\}$.

f) Longer than the 90-day warranty but less than 425 days.

Answer: $S = \{t: 90 < t < 425\}$

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Methods of Assigning Probability (1)

I. The *uniform* probability model - *equally likely* elementary outcomes
II. Probability as the *long-run relative frequency*

I. The uniform probability model

When the elementary outcomes are modeled as equally likely, we have a uniform probability model. If there are k elementary outcomes in $\mathcal{S}$, each is assigned the probability of $1/k$.

An event A consisting of m elementary outcomes is then assigned

$$P(A) = \frac{m}{k} = \frac{\text{No. of elementary outcomes in } A}{\text{No. of elementary outcomes in } \mathcal{S}}$$

Example: Find the probability of getting exactly one head in two tosses of a **fair coin**.

$S = \{HH, HT, TH, TT\}$ - The very concept of a fair coin implies that the four elementary outcomes in S are equally likely, assign the probability $\frac{1}{4}$ to each of them.

The event $A = [One\ head]$ has two elementary outcomes, $P(A) = 2/4 = \mathbf{0.5}$

Methods of Assigning Probability (2)

Probability as Long-Run Relative Frequency

We define $P(A)$, the probability of an event A, as the value to which the relative frequency stabilizes with increasing number of trials.

Although we will never know $P(A)$ exactly, it can be estimated accurately by repeating the experiment many times.

<u>Example</u>: If A denotes the event of a *birth on the weekend* (Saturday or Sunday), each newborn in country X can be considered as a trial of the experiment where the day of birth determines whether or not the event A occurs. One year, out of 4131 newborns 812 was born on weekend. *Estimate the probability that a baby will be born during a weekday.*

Solution: $P(weekday) + P(weekend) = 1, \quad P(\boldsymbol{weekend}) = \frac{812}{4131} = \mathbf{0.197}$

$$P(\boldsymbol{weekday}) = 1 - 0.197 = \mathbf{0.803}$$

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Event *Operations* and *Laws* of Probability

- *Three* most basic event relations:
  - Complement,
  - Union, and
  - Intersection
- *Two* laws of probability
  - Law of Complement,
  - Addition Law
    - Special addition law for incompatible events

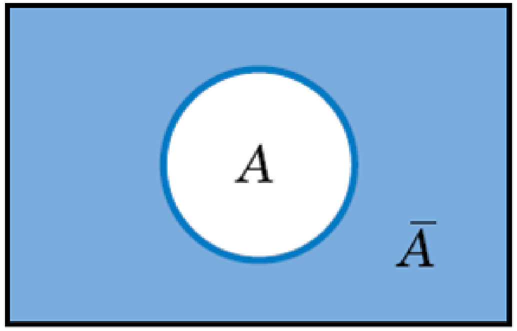
Event Operations (1)

- The **complement** of an event A, denoted $\bar{A}$ by is the set of all elementary outcomes that are not in A. The occurrence of $\bar{A}$ means that A ***does not occur***.

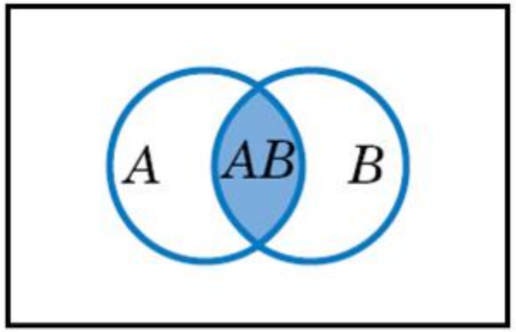
- The **union** of two events A and B, denoted by $A \cup B$, is the set of all elementary outcomes that are in A, B, or both. The occurrence of $A \cup B$ means that ***either A or B or both occur***.

- The **intersection** of two events A and B, denoted by AB, is the set of all elementary outcomes that are in A and B. The occurrence of AB means that ***both A and B occur.***

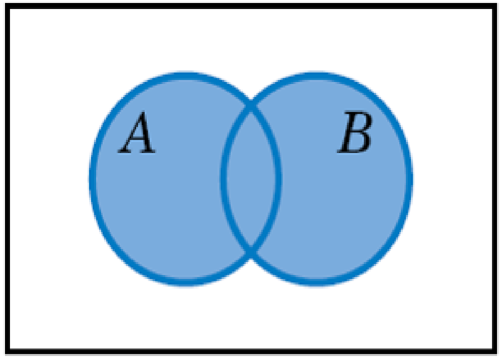
- Two events A and B are called **incompatible** or **mutually exclusive** if their intersection AB is empty.



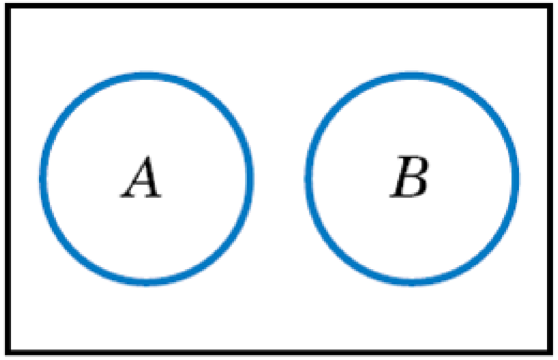
Complement $\bar{A}$



Intersection AB



Union $A \cup B$



Incompatible events

Event Operations (2)

Example: Four young lab puppies from different litters are available for a new method of training.

The possible choices of a pair of numbers from {1, 2, 3, 4} are listed and labeled as

$\{1, 2\}\ \ (e_1)\ \ \{2, 3\}\ \ (e_4\ )$
$\{1, 3\}\ \ (e_2)\ \ \{2, 4\}\ \ (e_5\ )$
$\{1, 4\}\ \ (e_3)\ \ \{3, 4\}\ \ (e_6\ )$

Dog	Sex	Age (weeks)
1	M	10
2	M	15
3	F	10
4	F	10

Let $A\ = [Same\ sex], B\ =\ [Same\ age], and\ C\ =\ [Different\ sexes]$. Give the compositions of the events find C, $\bar{A}$, $A \cup B$, AB, BC

Answers

$A = \{e_1, e_6\}$

$B = \{e_2, e_3, e_6\}$

$C = \{e_2, e_3, e_4, e_5\}$

$A{\cup}B = \{e_1, e_2, e_3, , e_6\}$

$AB = \{e_6\}$

$BC = \{e_2, e_3\}$

The event $\bar{A}$ is the same as the event C

Laws of Probability (1)

Law of Complement

$P(A) = 1 - P(\overline{A})$

$P(A) + P(\bar{A}) = 1 = P(S)$

Addition Law

$P(A \cup B) = P(A) + P(B) - P(AB)$

subtract $P(AB)$ from $P(A) + P(B)$ to avoid double counting

Special Addition Law for Incompatible Events

$P(A \cup B) = P(A) + P(B)$

the events A and B are incompatible, their intersection AB is empty, so $P(AB) = 0$

Laws of Probability (2)

<u>Example</u>: Sample space consists of 8 elementary outcomes with the following probabilities.

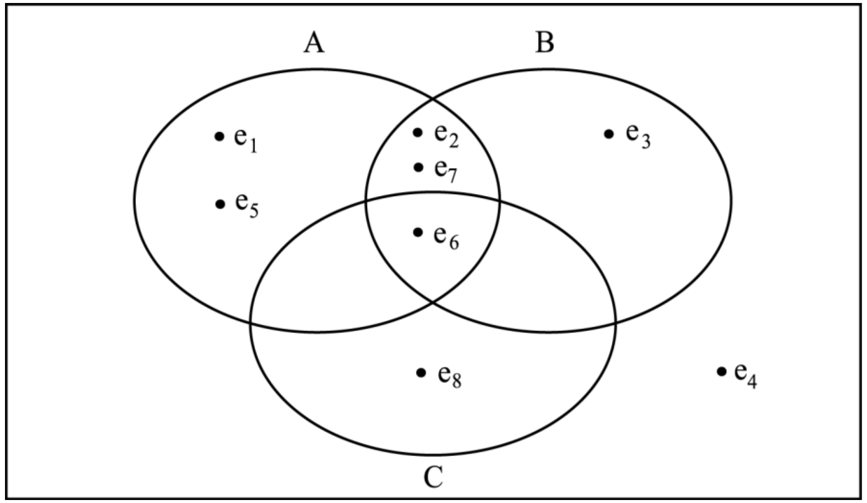
$P(e_1)$= 0.07 $\quad$ $P\ (e_2)$= $P(e_3)$= $P(e_4)$ $= 0.11$

$P(e_5\ )$= $P(e_6\ )$= $P(e_7)$= $P(e_8)$ $= 0.15$

Three events are given as A={e_1, e_2 , e_5 , e_6 , e_7}, B $= \{e_2, e_3, e_6, e_7\}$, $C = \{e_6, e_8\}$.

a) Draw a Venn diagram and show these events.

<u>Answer</u>



b) Give the composition, determine:

i. P($\bar{B}$)

Answer: $\bar{B}$ ={e_1, e_4 , e_5, e_8},

$P(B) = 0.07 + 0.11 + 0.15 + 0.15 = \mathbf{0.48}$

ii. P(BC)

Answer: $BC = \{e_6\}$, $\quad P(BC) = \mathbf{0.15}$

iii. P($A \cup C$**)**

***Answer*:** $A \cup C = \{e_1, e_2, e_5, e_6, e_7, e_8\}$

P($A \cup C$**)** = 0.07 + 0.11+ 0.15+ 0.15+ 0.15+ 0.15

= **0.78**

iv. P($\bar{A} \cup C$**)**

Answer: $\bar{A} \cup C$={e_3, e_4, e_6, e_8}, P($\bar{A} \cup C$**)** = 0.11+ 0.11+ 0.15+ 0.15 = **0.70**

Laws of Probability (3)

Example…cont'ed

v. *C* does not occur.

Answe*r*: $\bar{C}$ $=\{e_1, e_2, e_3, e_4, e_5, e_7\}$, $\mathrm{P}(\bar{C}) = 0.07 + 0.11 + 0.11 + 0.11 + 0.15 + 0.15 = \mathbf{0.70}$

vi. Both *A* and *B* occur.

Answer: $AB = \{e_2, e_6, e_7\}$, $P(AB) = 0.11 + 0.15 + 0.15 = \mathbf{0.41}$

vii. *A* occurs and *B* does not occur

Answer: $A\bar{B} = \{e_1, e_5\}$, $P(A\bar{B}) = 0.07 + 0.15 = \mathbf{0.22}$

viii. Neither *A* nor *C* occurs

Answer: $\bar{A}\bar{C} = \{e_3, e_4\}$, $P(\bar{A}\bar{C}) = 0.11 + 0.11 = \mathbf{0.22}$

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Conditional Probability (1)

The **conditional probability** of A given B is denoted by $P(A|B)$ and defined by the formula

$$P(A|B) = \frac{P(AB)}{P(B)}$$

Equivalently, this formula can be written

$$P(AB) = P(B)P(A|B)$$

This latter version is called the **multiplication law of probability.**

<u>Example</u>: There are 25 pens in a container on your desk. Among them, 20 will write well but 5 have defective ink cartridges. You will select 2 pens to take to a business appointment. Calculate the probability that:

a) Both pens are defective.
b) One pen is defective but the other will write well.

Answer

a) Lets use D for "defective" and G for "writes well". Using the multiplication law, we write

$$P(D_2D_1) = P\ (D_1)P(D_2|D_1)$$

Conditional Probability (2)

Example…cont'ed

$P(D_1) = \frac{5}{25}$, $P(D_2|D_1) = \frac{4}{24}$ $P(\text{ } both\ defective) = P\ (D_2 D_1) = \frac{5}{25} \text{x} \frac{4}{24} = \frac{1}{30} = \mathbf{0.033}$

b) The event [exactly one defective] is the union of the *two incompatible events* $G_1 D_2$ and $D_1 G_2$

$$P\ (G_1 D_2) = \frac{20}{25} \text{x} \frac{5}{24} = \frac{1}{6}$$

$$P\ (D_1 G_2) = \frac{5}{25} \text{x} \frac{20}{24} = \frac{1}{6}$$

The required probability is $P\ (G_1 D_2) + P\ (D_1 G_2) = \frac{1}{6} + \frac{1}{6} = \frac{2}{6} = \mathbf{0.333}$

Independence (1)

- The events are independent if the occurrence of B *has no bearing* on the assessment of the probability of A.

Two events A and B are **independent** if

$$P(A|B) = P(A)$$

Equivalent conditions are

$$P(B|A) = P(B)$$

or

$$P(AB) = P(A)P(B)$$

Recall that $P(A|B) = P(AB)/P(B)$, so that

$P(A|B) = P(A)$ is equivalent to

$$P(AB) = P(A)P(B)$$

$$P(B|A) = \frac{P(AB)}{P(A)} = \frac{P(A)P(B)}{P(A)} = P(B)$$

The form $P(AB) = P(A)P(B)$ shows that the definition of independence is symmetric in A and B.

Independence (2)

Example: An urn contains *two green* balls and *three red* balls. Suppose *two balls* will be drawn at random *one after another* and *without replacement* (i.e., the first ball drawn is *not* returned to the urn before the second one is drawn).

a) Find the probabilities of the events

$A =$ [Green ball appears in the first draw]

$B =$ [Green ball appears in the second draw]

2	Green
3	Red
5	

$\rightarrow 2$, without replacement

Answer: Lets denote *G* for green, *R* for red. Since the event *A*, a green ball appears in the first draw, has nothing to do with the second draw, lets identify $A = G_1$ so,

$$P(A) = P(G_1) = \frac{2}{5} = \mathbf{0.4}$$

The event B = G_2 is the union of G_1G_2 and R_1G_2

$$P(G_1G_2) = P(G_1)P(G_2|G_1) = \frac{2}{5} x \frac{1}{4} = \frac{2}{20}$$

$$P(R_1G_2) = P(R_1)P(G_2|R_1) = \frac{3}{5} x \frac{2}{4} = \frac{6}{20}$$

$$P(B) = P(G_2) = \frac{2}{20} + \frac{6}{20} = \frac{8}{20} = \mathbf{0.4}$$

Independence (2)

Example…cont'ed

b) Are the two events independent? Why or why not?

Answer: $P(AB) = P(G_1G_2) = \frac{2}{20} = 0.1$ On the other hand, $P(A)P(B) = 0.4x0.4 = 0.16$ and this is different from $P(AB)$. Therefore, A and B are *not independent*.

c) Now suppose two balls will be drawn *with replacement* (i.e., the first ball drawn will be returned to the urn before the second draw). Repeat parts (a) and (b).

Answer: Now, lets redo parts (a) and (b) *with* replacement:

Part (a)

$$P(A) = P(G_1) = \frac{2}{5} = \mathbf{0.4}$$

$$P(G_1G_2) = P(G_1)P(G_2|G_1) = \frac{2}{5}x\frac{2}{5} = \frac{4}{25}$$

$$P(R_1G_2) = P(R_1)P(G_2|R_1) = \frac{3}{5}x\frac{2}{5} = \frac{6}{25}$$

$$P(B) = P(G_2) = \frac{4}{25} + \frac{6}{25} = \frac{10}{25} = \mathbf{0.4}$$

Part (b)

$$P(AB) = P(G_1G_2) = \frac{4}{25} = 0.16$$

$$P(A)P(B) = 0.4x0.4 = 0.16 = P(AB)$$

Since these probabilities are equal, the *events are independent*.

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Bayes' Theorem (1)

- Bayes' Theorem expresses the conditional probability $P(A|B)$ in terms of $P(B|A)$ and the two *unconditional* probabilities P(A) and P(B).
- Before present the details of Bayes' Theorem, lets introduce the concept *rule of total probability*.
- An event A can occur either when an *event* B *occurs* or when it *does not occur*. That is, A can be written as the ***disjoint union*** of AB and $A\bar{B}$.

$P(A) = P(AB) + P(A\bar{B})$ Using multiplication rule of probability ⟹

Rule of Total Probability

$$P(A) = P(A \mid B)P(B) + P(A \mid \bar{B})P(\bar{B})$$

Example: Let A be the event that a person *tests positive* for a serious virus and B be the event that the person *actually has the virus*. Suppose that the virus is present in 1.4% of the population. Suppose the conditional probability that the test is positive, given that the person has the virus, is $0.995 = P(A|B)$. Also suppose that $P(A|\bar{B}) = 0.02$ is the conditional probability that a person not having the virus tests positive; a false positive. Determine the probability that a person will test positive, P(A).

Answer: $P(B)$ =0.014 so, 1- $P(B) = P(\bar{B}) = 0.986$ then,

$$P(A) = P(A|B)\text{P(B)}+ P(A|\bar{B})P(\bar{B}) = 0.995x0.014 + 0.02x0.986 = \mathbf{0.034}$$

Bayes' Theorem (2)

- We know the probability $P(B)$, so $P(\bar{B}) = 1 - P(B)$ which is called **prior probabilities** since they represent the probabilities associated with B and $\bar{B}$ before we know the status of event A or any other event.
- When we know the two conditional probabilities $P(A|B)$ and $P(A|\bar{B})$, the probability of B can be updated observing the status of A. the updated or **posterior probability** of B is given by the conditional probability.

$P(B|A) = \frac{P(AB)}{P(A)}$ where, $P(AB) = P(A|B)P(B), P(A) = P(A|B)\mathrm{P(B)} + P(A|\bar{B})P(\bar{B})$. Substituting these two alternate expressions into the formula for conditional probability, we obtain ***Bayes' Theorem***.

Bayes Theorem

$$P(B \mid A) = \frac{P(A \mid B)P(B)}{P(A \mid B)P(B) + P(A \mid \bar{B})P(\bar{B})}$$

The posterior probability of $\bar{B}$ is then $P(\bar{B} \mid A) = 1 - P(B \mid A)$.

Bayes' Theorem (3)

Example: A first step toward identifying spam is to create a list of words that are more likely to appear in spam than in normal messages. Suppose a specified list of words is available and that your database of 1000 messages contains 350 that are spam. Among spam messages, 280 contain words in the list. Of 650 normal messages, only 65 contains words in the list.

a) Obtain the probability that a message is spam given that the message contains words in the list.

Answer:

let $A = [messages\ contains\ words\ in\ list]$ an let $B = [message\ is\ spam]$

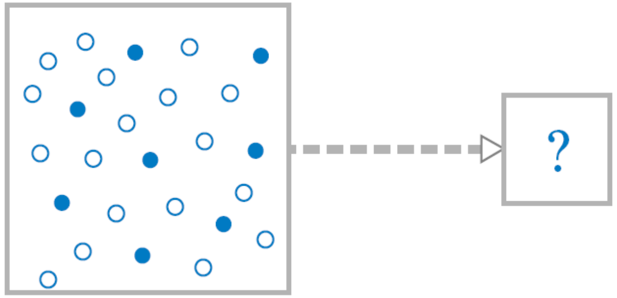
P(B)=$\frac{350}{1000} = 0.35$, $P(\bar{B}) = 1 - 0.35 = 0.65$, $P(A|B) = \frac{280}{350} = 0.80$, $P(A|\bar{B}) = \frac{65}{650} = 0.10$.

Bayes' Theorem expresses the probability of being spam, given that a message is identified as spam, as

$$P(B|A) = \frac{P(A|B)P(B)}{P(A|B)\mathrm{P(B)} + P(A|\bar{B})P(\bar{B})}$$

the updated or posterior probability, $P(A|B) = \frac{0.80x0.35}{0.80x0.35 + 0.10x0.65} = \frac{0.28}{0.345} = \mathbf{0.812.}$

Conclusion



A probability problem asks:

“What is the probability that the sample will have ...?

Statistical inference asks:

“What models of the population (blackbox) make the observed sample plausible?