Solution - IV

Question 1

a) The table below lists the 16 possible samples (x_1, x_2), along with the corresponding values of $\bar{X}$. Since n = 2 , the sample mean for each member of the sample is calculated using the formula $\bar{X} = \frac{x_1+x_2}{2}$

$(\boldsymbol{x_1}, \boldsymbol{x_2})$	(0,0)	(0,2)	(0,4)	(0,6)	(2,0)	(2,2)	(2,4)	(2,6)
$\bar{\boldsymbol{X}}$	0	1	2	3	1	2	3	4

$(\boldsymbol{x_1}, \boldsymbol{x_2})$	(4,0)	(4,2)	(4,4)	(4,6)	(6,0)	(6,2)	(6,4)	(6,6)
$\bar{\boldsymbol{X}}$	2	3	4	5	3	4	5	6

b) The 16 possible samples are equally likely, so each has a probability $1/16$ of occurring. The sampling distribution of $\bar{X}$ is obtained by listing the distinct values of $\bar{X}$ along with the corresponding probabilities, as follows:

$\bar{\boldsymbol{X}}$	Probability
0	$1/16$
1	$2/16$
2	$3/16$
3	$4/16$
4	$3/16$
5	$2/16$
6	$1/16$
Total	1

c) Since each of the four values of X are equally likely, each has probability of ¼ of occurring. The probability distribution is tabulated below, along with other calculations needed to compute the population mean and standard deviation.

x	$f(x)$	$xf(x)$	$x^2f(x)$
0	$1/4$	0	0
2	$1/4$	$2/4$	$4/4$
4	$1/4$	$4/4$	$16/4$
6	$1/4$	$6/4$	$36/4$
Total	1	$12/4$	$56/4$

Using the values in the table, we have the following:

$$\mu = \sum xf(x) = 12/4 = 3$$

$$\sigma^2 = E(X^2) - \mu^2 = \sum x^2 f(x) - \mu^2 = 56/4 - 3^2 = 5, \quad \text{so that } \sigma = \sqrt{5}$$

d) For $n = 2$, we know that the mean and standard deviation of the sampling distribution of $\bar{X}$ must be as follows:

$$E(\bar{X}) = \mu = \mathbf{3}$$

$$sd(\bar{X}) = \frac{\sigma}{\sqrt{2}} = \frac{\sqrt{5}}{\sqrt{2}} = \sqrt{\frac{\mathbf{5}}{\mathbf{2}}}$$

We verify these by actually calculating the distribution of $\bar{X}$:

$\bar{X}$	$f(\bar{X})$	$\bar{X}f(\bar{X})$	$\bar{X}^2 f(\bar{X})$
0	$1/16$	0	0
1	$2/16$	$2/16$	$2/16$
2	$3/16$	$6/16$	$12/16$
3	$4/16$	$12/16$	$36/16$
4	$3/16$	$12/16$	$48/16$
5	$2/16$	$10/16$	$50/16$
6	$1/16$	$6/16$	$36/16$
Total	1	$48/16$	$184/16$

Using the values in the table, we have the following (which do indeed confirm the above assertion):

$$E(\bar{X}) = \sum \bar{X}f(\bar{X}) = 48/16 = \mathbf{3}$$

$$Var(\bar{X}) = E(\bar{X}^2) - (E(\bar{X}))^2 = \sum \bar{X}^2 f(\bar{X}) - (E(\bar{X}))^2 = 184/16 - 3^2 = 5/2$$

$$sd(\bar{X}) = \sqrt{\frac{\mathbf{5}}{\mathbf{2}}}$$

Question 2

We have $\mu = 21.1$, $\sigma = 2.6$, and $n = 150$

a) We have $E(\bar{X}) = \mu = 21.1$ and $sd(\bar{X}) = \frac{\sigma}{\sqrt{n}} = \frac{2.6}{\sqrt{150}} = 0.2123$. Since $n = 150$ is large, the central limit theorem ensures that the distribution of $\bar{X}$ is approximately normal with mean and standard deviation as calculated above.

b) The standardized variable is $Z = \frac{\bar{X}-21.1}{0.2123}$. As such, we have

$$P[17.85 < \bar{X} < 25.65] = P\left[\frac{17.85 - 21.1}{0.2123} < Z < \frac{25.65 - 21.1}{0.2123}\right]$$
$$= P[-15.31 < Z < 21.43] = \mathbf{1}$$

c) $P[\bar{X} < 25.91] = P\left[Z > \frac{25.91-21.1}{0.2123}\right] = P[Z > 22.66] = \mathbf{0}$

Question 3

The standard deviation of $\bar{X}$ is $\frac{\sigma}{\sqrt{n}}$, where σ is the population standard deviation.

a) In order to have $\frac{\sigma}{\sqrt{n}} = \frac{\sigma}{4}$, we require that $\sqrt{n} = 4$, or $n = 16$

b) In order to have $\frac{\sigma}{\sqrt{n}} = \frac{\sigma}{7}$, we require that $\sqrt{n} = 7$, or $n = 49$

c) In order to have $\frac{\sigma}{\sqrt{n}} = (0.12)\sigma$, we require that $\sqrt{n} = \frac{\sigma}{0.12}$ so that $n = (\frac{1}{0.12})^2 = 69.44$.

Since the sample size must be an integer value, we would use $n = 70$ to be conservative.

Question 4

Since $n = 49$ is large, the distribution of $\bar{X}$ is approximately normal with mean $= \mu$ (the population mean), and $sd(\bar{X}) = \frac{\sigma}{\sqrt{n}} = \frac{21}{\sqrt{49}} = 3$. Hence, $Z = \frac{X-\mu}{3}$ is approximately standard normal.

a) $P[-2 < \bar{X} - \mu < 2] = P\left[-\frac{2}{3} < Z < \frac{2}{3}\right] = P\left[Z < \frac{2}{3}\right] - \left[Z < -\frac{2}{3}\right]$
$$= 0.7470 - 0.2530 = \mathbf{0.494}$$

b) Since $P[-1.645 < Z < 1.645] = 0.90$, the number k must be $1.645(sd(\bar{X}))$
Hence, $k = 1.645(3) = 4.935$.

c) $P[|X - \mu| > 4] = P\left[|Z| > \frac{4}{3}\right] = P[|Z| > 1.33] = 2P[|Z| < 1.33] = 2(0.0918) = \mathbf{0.1836}$